

Cecotti–Fendley–Intriligator–Vafa Index in a Box

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Abstract

The Cecotti–Fendley–Intriligator–Vafa (CFIV) index in two-dimensional $\mathcal{N} = (2, 2)$ models is revisited. We address the problem of “elementary” (nontopological) excitations over a kink solution, in the one-kink sector (using the Wess–Zumino or Landau–Ginzburg models with two vacua as examples). In other words, we limit ourselves to the large- β limit. The excitation spectrum over the BPS-saturated at the classical level kink is discretized through a large box with judiciously chosen boundary conditions. The boundary conditions are designed in such a way that half of supersymmetry is preserved as well as the BPS kink itself, and relevant zero modes. The excitation spectrum acquires a mass gap. All (discretized) excited states enter in four-dimensional multiplets (two bosonic states + two fermionic). Their contribution to ind_{CFIV} vanishes level by level. The ground state contribution produces $|\text{ind}_{\text{CFIV}}| = 1$.

1 Introduction

In 1982 Witten suggested [1] in supersymmetric theories his famous index

$$\mathrm{Tr}(-1)^F \equiv \sum_n \langle n | (-1)^F | n \rangle, \quad (1)$$

where F is the fermion number operator. Sometimes the Witten index is represented in the form

$$\mathrm{ind}_W = \mathrm{Tr}(-1)^F e^{-H\beta}, \quad (2)$$

where the last factor is introduced for ultraviolet (UV) regularization. If the supersymmetric theory under consideration is properly regularized (i.e. there are no gapless excitations and supersymmetry is unbroken) ind_W does not depend on β since all nonzero-energy states cancel out because of the Bose–Fermi degeneracy of the spectrum. This can be achieved on compact spaces, generally speaking. Then Eq. (2) can be replaced by the original $\mathrm{Tr}(-1)^F$.

The crucial feature of the Witten index is that it is invariant under continuous deformations of the initial theory (see Section 7 below for a more detailed discussion). One can deform the theory in any continuous way, in particular, in a way, that makes Witten index calculation easy. The Witten index counts the difference between the number of bosonic and fermionic vacua. Therefore, if $\mathrm{ind}_W \neq 0$ one can conclude that supersymmetry is not spontaneously broken. In essence, in four-dimensional theories with a mass gap ind_W gives the number of supersymmetric vacua. There exists a vast literature devoted to Witten index in various supersymmetric theories.

A related “index” was suggested by Cecotti, Fendley, Intriligator, and Vafa (CFIV) [2]. The CFIV “index” plays the same role with regard to short soliton (kink) multiplets in $\mathcal{N} = 2$ two-dimensional models as the Witten index for supersymmetric vacua. The quotation marks are used to emphasize that the “index” does not depend only on a limited class of continuous deformations, as will be explained shortly.¹ It was introduced as follows:

$$\mathrm{ind}_{\mathrm{CFIV}} = \mathrm{Tr} \left[F (-1)^F e^{-H\beta} \right]. \quad (3)$$

For any long supermultiplet (i.e. the supermultiplet which is not BPS-saturated)

$$\mathrm{ind}_{\mathrm{CFIV}} \supset f(-1)^f + 2(f+1)(-1)^{f+1} + (f+2)(-1)^{f+2} = 0. \quad (4)$$

¹Hereafter the quotation marks will be omitted.

The fermion charges of the states in the long multiplet are $f + 2$, $f + 1$, and f , respectively, while the corresponding state multiplicities can be read off from (4).

However, the short (BPS-saturated) supermultiplets, if present in the theory, produce a nonvanishing contribution,

$$\text{ind}_{\text{CFIV}} \supset f(-1)^f + (f + 1)(-1)^{f+1} = (-1)^{f+1}. \quad (5)$$

Thus, the value of the CFIV index counts the number of short multiplets that cannot be combined into long ones. Again, if $\text{ind}_{\text{CFIV}} \neq 0$ is established in an appropriately deformed theory, one can be certain that the original (undeformed theory) supports BPS-saturated solitons.

The CVIF index does *not* depend on continuous deformations of the D terms (i.e. changing the Kähler potential). Unlike the Witten index, the CVIF index can depend, however, on deformations of the F terms [2] (see Section 7 for a more detailed discussion). That's why in fact it is not index in the conventional sense, unlike the Witten index, and that's why we, following the authors, used the quotation marks in the discussion preceding Eq. (3). Nevertheless, the CVIF index is a useful tool, since in some important instances one needs to explore the issue of the BPS saturation under the condition that the superpotential is exactly known while the Kähler potential is not. Then, calculating the CVIF index, say, for the canonic Kähler potential we may be sure that it stays the same for any other Kähler potential that can be obtained from the canonic one by a continuous deformation.

The regularizing factor $e^{-H\beta}$ is introduced in (3) for the same reason as in (2). In the presence of a continuous spectrum of excitations, isolating and counting distinct supermultiplet contributions to the indices is a subtle procedure. The notion of degeneracy of the bosonic and fermionic states, if both belong to the continuous spectra, is not well defined because one should include the density of these states in the count. It is often said that nonzero-energy states in (2) and excitations over the BPS kink in (3) can defy the Bose–Fermi cancellation and produce nonvanishing contributions to the indices due to lack of supersymmetry in the density of the excited states.

In the pioneering paper [2] the main emphasis was put on the the multikink sectors. At large β they are exponentially suppressed compared to the one-kink sector, but at finite β their contribution is highly nontrivial. In fact, the authors of [2] succeeded in obtaining and solving an exact equation for ind_{CFIV} as a function of β . A number of models considered in [2] are integrable, implying, through the thermodynamic Bethe ansatz, an exact S matrix. This knowledge led the authors to the exact solution.

The problem we address is more limited in scope; it concerns the infrared limit $\beta \rightarrow \infty$. In other words we will focus on the one-kink sector, with the goal of obtaining a clear-cut physical interpretation of the role of “elementary” (nontopological) excitations over a kink solution in the process of calculating ind_{CFIV} in the infrared limit.

Problems arising due to the continuous nature of the spectrum of elementary excitations is not a unique feature of the CFIV index. The presence of continuum presents a certain difficulty in the calculation of the Witten index too (see e.g. [3]). A physical way to get rid of this subtlety is to discretize the would-be continuous spectrum of excitations. This can be viewed as a technical problem, of course, but its analysis is helpful in understanding related issues (such as curves of the marginal stability, or ‘domain walls’).

We suggest an infrared regularization (a large box regularization) which discretizes the spectrum of excitations over the kink under consideration and maintains enough supersymmetry to provide a level-by-level cancellation so that the ind_{CFIV} is saturated by the ground-state supermultiplet. A very clear-cut physical picture behind the CFIV index calculation emerges.

Before we proceed to the outline of our main idea we hasten to make a reservation. Since the total spatial momentum is conserved, we can (and will) calculate ind_{CFIV} in the subspace of the Hilbert space with the vanishing total spatial momentum. The authors of [2] chose not to use this possibility, and integrated over kink’s spatial momentum in calculating ind_{CFIV} . Correspondingly, in defining ind_{CFIV} , they had to use appropriate normalization factors which will be omitted in our analysis. We do not need them since we do not integrate over kink’s spatial momentum. In the limit $\beta \rightarrow \infty$ the one-kink state at rest is singled out as the state with the lowest energy for the given boundary conditions.

Our idea is as follows.² Assume that the spatial dimension is limited by a large box, with size L and appropriately chosen boundary conditions. In a sense, the space is compactified, but at the very end we can tend $L \rightarrow \infty$. The boundary conditions on the edges of the large box must be imposed in such a way that:

- (i) they discretize the excitation spectrum;
- (ii) they maintain half supersymmetry; and
- (iii) they preserve both, the soliton under consideration and fermionic zero modes, which correspond to the broken supercharges.

²The basic idea presented here is a generalization of the consideration carried out previously [4] in the context of $\mathcal{N} = (1, 1)$ theories.

If this task is achieved, then only the ground state in the given sector will contribute to (3). Correspondingly, we obtain (4) and (5). For all non-BPS (excited) levels the degeneracy is four-fold. Indeed, two preserved supercharges guarantee doubling of all nonzero energy levels, while the addition of the fermionic zero mode does not change the energy. Denoting by c_0^\dagger and q^\dagger the creation operator for the fermionic zero mode and the preserved supercharge, respectively, the long multiplet can be written as

$$|n\rangle, q^\dagger |n\rangle, c_0^\dagger |n\rangle, c_0^\dagger q^\dagger |n\rangle, \quad (6)$$

while the short multiplet is

$$|0\rangle, c_0^\dagger |0\rangle. \quad (7)$$

One can see that the fermion numbers for such states are indeed as in (4). As a result, no dependence on β will ever appear.

Admittedly, conventional choices of the boundary conditions destroy supersymmetry, and with it is gone degeneracy between individual bosonic and fermionic excitation energies. Moreover, “inappropriate” boundary conditions may destroy the kink itself. The straightforward intuitive counting of ind_{CFIV} , as in (3) and (4), becomes impossible, and one has to invoke the original CFIV procedure [2] or similar, which entangles the continuous spectrum.

In this paper we will show that the boundary conditions satisfying the above conditions (i), (ii) and (iii) exist in two-dimensional $\mathcal{N} = (2, 2)$ models – the subject of the original CFIV analysis – in much the same way as they had been shown to exist [4] in two-dimensional $\mathcal{N} = (1, 1)$ models.

Below we will consider as an example the Wess–Zumino models with one or more superfields. We will put the system in a large box, preserving the soliton solution, two supercharges, and the fermionic zero modes. As a result, the ground states form a short supermultiplet, while all (discretized) excited states are in the long supermultiplets. This allows us to isolate contributions to the CFIV index at each energy level separately.

Why the knowledge of indices is so useful, in particular, the CFIV index? There exists a number of important problems in which the precise form of the Kähler potential is not known. For instance, in the $\text{CP}(N - 1)$ models the mirror representation exists [5] which allows one to establish the superpotential, but not the Kähler potential. Assume we want to address the question whether BPS-saturated solitons exist in the $\text{CP}(N - 1)$ models. No direct solution for solitons is possible because of strong coupling in the $\text{CP}(N - 1)$ models. One can then turn to the mirror representation. Due to the fact that the CFIV index is

independent of the Kähler potential, one can use the canonic Kähler potential, determine the CFIV index, and make the conclusion of the existence of N BPS-saturated solitons. This conclusion will stay valid in the $CP(N-1)$ models.

In certain instances a relation between the CFIV index for kinks and Witten index for an emerging model on the kink world line can be established. The BPS saturation of the kink under consideration is then interpreted as the existence of the supersymmetric vacuum in quantum mechanics on the world line. The fact that $|\text{ind}_{\text{CFIV}}|$ is integer in our procedure has a clear-cut meaning from this standpoint.

Organization of the paper is as follows. In Section 2 we present the general idea as to how to put the system in the box satisfying the conditions listed above. In Sections 3 and 4 we describe the general $\mathcal{N} = 2$ system in quadratic approximation and show that the only nonzero contribution comes from the ground state. In Section 5 we consider some examples. In Section 6 we briefly describe generalization to multifield models.

2 General construction. Putting the system in a box

To begin with we will consider the system given by the Lagrangian of the form

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (\partial_R \bar{\phi} \partial_L \phi + \partial_L \bar{\phi} \partial_R \phi) + \bar{F} F + \mathcal{W}'(\phi) F + \bar{\mathcal{W}}'(\bar{\phi}) \bar{F} \\ & + i \bar{\psi}_R \partial_L \psi_R + i \bar{\psi}_L \partial_R \psi_L - i \mathcal{W}''(\phi) \psi_R \psi_L - i \bar{\mathcal{W}}''(\bar{\phi}) \bar{\psi}_R \bar{\psi}_L, \end{aligned} \quad (8)$$

invariant under the following $\mathcal{N} = (2, 2)$ SUSY transformations:

$$\begin{aligned} \delta \phi &= i\sqrt{2} (\varepsilon_L \psi_R + \varepsilon_R \psi_L), \\ \delta \psi_L &= -\sqrt{2} \bar{\varepsilon}_R \partial_L \phi - \sqrt{2} \varepsilon_L F, \\ \delta \psi_R &= -\sqrt{2} \bar{\varepsilon}_L \partial_R \phi + \sqrt{2} \varepsilon_R F, \\ \delta F &= i\sqrt{2} (\bar{\varepsilon}_L \partial_R \psi_L - \bar{\varepsilon}_R \partial_L \psi_R), \end{aligned} \quad (9)$$

where

$$\partial_{L,R} = \partial_0 \pm \partial_z. \quad (10)$$

If the superpotential \mathcal{W} has more than one minima there exist solitons interpolating between any two vacua at spatial infinities. Without loss of generality (see Appendix B) we can assume that

$$\gamma = \frac{\Delta \bar{\mathcal{W}}}{|\Delta \mathcal{W}|} = 1, \quad (11)$$

where $\Delta\bar{\mathcal{W}} = \bar{\mathcal{W}}(\infty) - \bar{\mathcal{W}}(-\infty)$. In this case the Bogomol'nyi equation for the BPS soliton is given by

$$\partial_z \phi_k - \bar{\mathcal{W}}'(\bar{\phi}_k) = 0, \quad (12)$$

where the subscript k stands for kink. A solution to this equation, if it exists, breaks two supercharges out of four. The action of the broken supercharges produces fermionic zero modes,

$$\begin{aligned} \delta\psi_L &= \sqrt{2}\eta\partial_z\phi_k, \\ \delta\psi_R &= \sqrt{2}\bar{\eta}\partial_z\phi_k. \end{aligned} \quad (13)$$

There are two real zero modes (or a complex mode and its conjugate).

The supercharge preserving the BPS solution corresponds to a specific linear combination of the $\mathcal{N} = (2, 2)$ supercharges, namely,

$$q = i\sqrt{2} \int dz [\psi_R (\partial_R \bar{\phi} + \mathcal{W}') + \bar{\psi}_L (\partial_L \phi - \bar{\mathcal{W}}')]. \quad (14)$$

Needless to say, q^\dagger is conserved too. In the model under consideration, in addition to supersymmetry, there exists a chiral U(1) symmetry

$$\begin{aligned} \psi'_R &= \psi_R e^{i\alpha}, \\ \psi'_L &= \psi_L e^{-i\alpha}, \end{aligned} \quad (15)$$

see (8). The generator corresponding to this symmetry is the fermion number operator which can be defined as

$$F = \int dz (\bar{\psi}_R \psi_R - \bar{\psi}_L \psi_L). \quad (16)$$

Separating the contribution of the central charge ($Z = \int dz \partial_z (\mathcal{W} + \bar{\mathcal{W}})$) the Hamiltonian H can be written as

$$\begin{aligned} H &= \int dz \mathcal{H}, \\ \mathcal{H} &= \partial_z (\mathcal{W} + \bar{\mathcal{W}}) \\ &+ \partial_0 \bar{\phi} \partial_0 \phi + (\partial_z \phi - \bar{\mathcal{W}}') (\partial_z \bar{\phi} - \mathcal{W}') \\ &+ \frac{i}{2} (\bar{\psi}_L \partial_z \psi_L - \bar{\psi}_R \partial_z \psi_R - \partial_z \bar{\psi}_L \psi_L + \partial_z \bar{\psi}_R \psi_R) + i\mathcal{W}'' \psi_R \psi_L + i\bar{\mathcal{W}}'' \bar{\psi}_R \bar{\psi}_L. \end{aligned} \quad (17)$$

Using the expression for relevant operators in terms of fields and canonic commutation relations one obtains the following algebra:

$$\begin{aligned}
[F, q] &= -q, \\
[F, q^\dagger] &= q^\dagger, \\
[F, H - Z] &= -\frac{i}{2} \int dz \partial_z (\bar{\psi}_R \psi_R + \bar{\psi}_L \psi_L), \\
[q, H - Z] &= -\frac{1}{2\sqrt{2}} \int dz \partial_z [\bar{\psi}_L (\partial_L \phi - \bar{\mathcal{W}}') - \psi_R (\partial_R \bar{\phi} + \mathcal{W}')] .
\end{aligned} \tag{18}$$

In order for the fermion operator F and the supercharge q to be conserved in the system in the box, one has to require the vanishing of all boundary terms in (18). It is evident that the following choice of boundary conditions meets this requirement:

$$\begin{aligned}
(\bar{a}, -a) \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0, \\
(\bar{a}, -a) \begin{pmatrix} \partial_L \phi - \bar{\mathcal{W}}' \\ \partial_R \bar{\phi} + \mathcal{W}' \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0,
\end{aligned} \tag{19}$$

where a is an arbitrary complex constant. In Appendix A.1 we illustrate how the above boundary conditions work in the topologically trivial vacuum (i.e. without the kink background).

There are four boundary conditions in (19) for the fermions as is required by the first-order differential equations. However, those are the conditions for only two out of four (real) linear combinations of fermions. In order to have boundary conditions for the rest of fermions and at the same time not to overdetermine the system, one has to impose additional boundary conditions which are dependent (through the equations of motion; see Appendix A.3). As a result we get the second set of boundary conditions for the fermion fields,

$$(\bar{a}, -a) \begin{pmatrix} \partial_z & -\bar{\mathcal{W}}'' \\ \mathcal{W}'' & -\partial_z \end{pmatrix} \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} \Big|_{\pm \frac{L}{2}} = 0. \tag{20}$$

3 Quadratic approximation and the spectrum

It should be noted that the boundary conditions introduced above make half supersymmetry manifest in all orders of perturbation theory. This is not the end of the story, however. As

was explained in Section 1, we need to preserve the fermionic zero mode as well. Here we will demonstrate that the boundary conditions of the form (19) and (20) satisfy this requirement in the quadratic approximation. The generalization for any order in perturbation theory is given in Appendix A.2.

Expanding the Hamiltonian around the BPS background we arrive at

$$\begin{aligned}
[\mathcal{H} - \partial_z (\mathcal{W} + \bar{\mathcal{W}})]_{\text{quad}} &= \frac{1}{2} (\partial_0 \bar{\chi}, \partial_0 \chi) \begin{pmatrix} \partial_0 \chi \\ \partial_0 \bar{\chi} \end{pmatrix} \\
&+ \frac{1}{2} (\bar{\chi}, \chi) P^2 \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} + \frac{1}{2} (\bar{\psi}_L, \psi_R) P \begin{pmatrix} \psi_L \\ \bar{\psi}_R \end{pmatrix} \\
&- \frac{1}{2} (\bar{\psi}_R, \psi_L) P \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} - \frac{i}{2} \partial_z \left[(\bar{\chi}, -\chi) P \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \right], \quad (21)
\end{aligned}$$

where $\phi \equiv \phi_k + \chi$, and

$$\begin{aligned}
P &= \begin{pmatrix} i\partial_z & -i\bar{\mathcal{W}}'' \\ i\mathcal{W}'' & -i\partial_z \end{pmatrix}, \quad P^\dagger = P \\
P^2 &= \begin{pmatrix} -\partial_z^2 + \bar{\mathcal{W}}''\mathcal{W}'' & \partial_z \bar{\mathcal{W}}'' \\ \partial_z \mathcal{W}'' & -\partial_z^2 + \bar{\mathcal{W}}''\mathcal{W}'' \end{pmatrix}. \quad (22)
\end{aligned}$$

The fermionic zero mode (13) which satisfies

$$P \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} = 0, \quad (23)$$

is preserved by the choice

$$a = \partial_z \phi_k \quad (24)$$

which implies the linearized boundary conditions

$$\begin{aligned}
(\partial_z \bar{\phi}_k, -\partial_z \phi_k) P \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0, \\
(\partial_z \bar{\phi}_k, -\partial_z \phi_k) \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0, \\
(\partial_z \bar{\phi}_k, -\partial_z \phi_k) P \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0, \\
(\partial_z \bar{\phi}_k, -\partial_z \phi_k) \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} \Big|_{\pm \frac{L}{2}} &= 0.
\end{aligned} \tag{25}$$

The geometrical meaning of the relations above is the following. If we consider the \mathbb{C}^2 space with coordinates (z_1, z_2) , then any ϕ_k solution defines a subspace (line) in it

$$\gamma_k = (\phi_k, \bar{\phi}_k).$$

As a result, the boundary conditions (25) are the orthogonality conditions (at the boundary $\pm L/2$) of the vector normal to γ_k and the fluctuations.

Now, we expand the fields in a series of eigenfunctions of the operator P^2

$$\begin{aligned}
\begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} &= \sum_{n,s} b_{ns} \begin{pmatrix} f_n^s \\ \bar{f}_n^s \end{pmatrix} \\
\begin{pmatrix} \psi_L \\ \bar{\psi}_R \end{pmatrix} &= \sum_{n,s} \xi_{ns} \begin{pmatrix} f_n^s \\ \bar{f}_n^s \end{pmatrix},
\end{aligned} \tag{26}$$

where the index n labels the level corresponding to the eigenvalue ω_n while $s = 1, 2$ labels the eigenstates of the operator P with positive and negative eigenvalues correspondingly. The functions f_n^s are such that

$$\begin{aligned}
\sum_{n,s} f_n^s(x) \bar{f}_n^s(y) &= \delta(x - y), \\
\int dz \bar{f}_n^s(z) f_m^r(z) &= \frac{1}{2} \delta_{nm} \delta_{sr}
\end{aligned} \tag{27}$$

and they satisfy the boundary conditions

$$(\partial_z \bar{\phi}_k, -\partial_z \phi_k) \begin{pmatrix} f_n^s \\ \bar{f}_n^s \end{pmatrix} \Big|_{\pm \frac{L}{2}} = (\partial_z \bar{\phi}_k, -\partial_z \phi_k) P \begin{pmatrix} f_n^s \\ \bar{f}_n^s \end{pmatrix} \Big|_{\pm \frac{L}{2}} = 0. \tag{28}$$

Plugging the expansion (26) to the Hamiltonian (21) one gets

$$H - Z = \frac{1}{2}b_0^2 + \frac{1}{2} \sum_{n \neq 0, s} \left(b_{ns}^2 + \omega_n^2 b_{ns}^2 \right) + \sum_{n \neq 0} \omega_n \left(\bar{\xi}_{n1} \xi_{n1} + \xi_{n2} \bar{\xi}_{n2} \right), \quad (29)$$

which upon the change of the variables

$$\begin{aligned} a_{ns} &= \frac{\omega_n^{1/2} b_{ns} - i \omega_n^{-1/2} \dot{b}_{ns}}{\sqrt{2}}, \\ \xi_{n1} &= c_{n1}, \\ \xi_{n2} &= c_{n2}^\dagger, \\ \xi_0 &= c_0^\dagger, \end{aligned} \quad (30)$$

gives

$$H - Z = \frac{1}{2}b_0^2 + \sum_{n \neq 0, s} \omega_n \left(a_{ns}^\dagger a_{ns} + c_{ns}^\dagger c_{ns} \right). \quad (31)$$

4 The CFIV index

4.1 Fermion charge

We start from the following remark. As one can see from the expression for the Hamiltonian (31), there is an additional doubling of energy levels (four operators a_{ns} and c_{ns}).

The expression for the fermion number operator in terms of creation-annihilation operators takes the form

$$F = \frac{1}{2} \left(c_0^\dagger c_0 - c_0 c_0^\dagger \right) + \sum_{n \neq 0} \left(c_{n2}^\dagger c_{n2} - c_{n1}^\dagger c_{n1} \right). \quad (32)$$

Therefore the fermions of type c_{n1} have charge -1 while those of type c_{n2} have charge 1 . The fermions produced by c_0^\dagger have half-integer fermion charge due to the charge fractionalization [7].

4.2 Index

Choosing an arbitrary excited (non-BPS) state

$$|n\rangle = a_{js}^\dagger c_{jr}^\dagger \dots |0\rangle, \quad (33)$$

one finds the long multiplet in the form³

$$|n\rangle, q^\dagger |n\rangle, c_0^\dagger |n\rangle, c_0^\dagger q^\dagger |n\rangle, \quad (34)$$

whose contribution to the CFIV index vanishes. The only nonzero contribution to the index is from the short multiplet,

$$|0\rangle, c_0^\dagger |0\rangle, \quad (35)$$

which gives

$$\text{ind}_{\text{CFIV}} = \left(\frac{1}{2}\right) (-1)^{1/2} + \left(-\frac{1}{2}\right) (-1)^{-1/2} = i \quad (36)$$

Note that any additional (i.e. not required by the preserved two supercharges) level doubling (at the quadratic level), as indicated in the previous subsection, makes the level in question effectively $\mathcal{N} = (2, 2)$ supersymmetric. Four-dimensional multiplet (34) is accompanied by another four-dimensional multiplet with the same energy. The contribution to the index from such multiplets vanishes automatically. However, it is not clear whether or not this latter doubling persists in higher orders.

Moreover, even if it persists, the “other” long multiplets (other than (34)), taken individually, *do* contribute to the index see Eq. (47). This is because they are not genuine supermultiplets: the fermion charges defy Eq. (4).

5 Examples

5.1 Superpolynomial model

For the polynomial superpotential with real coefficients

$$\mathcal{W}(\Phi) = \frac{m^2}{4\lambda} \Phi - \frac{\lambda}{3} \Phi^3, \quad (37)$$

the BPS kink solution is given by the following expression

$$\phi_k(z) = \frac{m}{2\lambda} \tanh \frac{m}{2} z. \quad (38)$$

³We assume the q^\dagger does not annihilate the state $|n\rangle$.

It is purely real. The Hamiltonian in quadratic approximation can be written as follows:⁴

$$\begin{aligned} [\mathcal{H} - \partial_z (\mathcal{W} + \bar{\mathcal{W}})]_{\text{quad}} &= \partial_0 \chi_1 \partial_0 \chi_1 + \partial_0 \chi_2 \partial_0 \chi_2 \\ &+ \chi_1 p^\dagger p \chi_1 + i v_2 p u_2 + i u_1 p v_1 + \chi_2 p p^\dagger \chi_2 - i v_1 p^\dagger u_1 - i u_2 p^\dagger v_2, \end{aligned} \quad (39)$$

where the operator p is defined by

$$p = \partial_z - \mathcal{W}''(\phi_k) = \partial_z + 2\lambda\phi_k, \quad (40)$$

and the boundary conditions (25) take the form

$$p\chi_1 \Big|_{z=\pm\frac{L}{2}} = u_1 \Big|_{z=\pm\frac{L}{2}} = v_2 \Big|_{z=\pm\frac{L}{2}} = \chi_2 \Big|_{z=\pm\frac{L}{2}} = pu_2 \Big|_{z=\pm\frac{L}{2}} = pv_1 \Big|_{z=\pm\frac{L}{2}} = 0. \quad (41)$$

The operators $p^\dagger p$ and pp^\dagger have the same eigenvalues (except zero) and their eigenfunctions are related by⁵

$$\begin{aligned} \tilde{f}_n &= \frac{1}{\omega_n} p f_n, \\ f_n &= \frac{1}{\omega_n} p^\dagger \tilde{f}_n, \end{aligned} \quad (42)$$

except for the zero mode of the operator $p^\dagger p$,

$$p f_0 = 0. \quad (43)$$

The expansion in series in eigenfunctions leads to the same result as described above.

⁴The following change of the variables was performed

$$\begin{aligned} \chi &= \chi_1 + i\chi_2, \\ \psi_R &= \frac{\psi_2 - \psi_1}{\sqrt{2}}, \quad \psi_1 = u_1 + iu_2, \\ \psi_L &= \frac{\psi_2 + \psi_1}{\sqrt{2}}, \quad \psi_2 = v_1 + iv_2. \end{aligned}$$

⁵The modes satisfy the boundary conditions $\tilde{f} \Big|_{z=\pm\frac{L}{2}} = 0$ and $pf \Big|_{z=\pm\frac{L}{2}} = 0$.

5.2 CP(1) mirror

Another example is the system with the superpotential appearing as a mirror in the CP(1) model [5]

$$\mathcal{W} = \frac{\lambda}{2} \left(\Phi + \frac{v^2}{\Phi} \right). \quad (44)$$

There are two BPS kinks in this case corresponding to two semicircles, in the upper and lower complex half planes,

$$\phi_k^{1,2} = v \left(\tanh \frac{\lambda z}{v} \pm i \cosh^{-1} \frac{\lambda z}{v} \right). \quad (45)$$

Then we repeat consideration of the previous subsection.

5.3 C conjugation

For systems such as that described by the superpotential (37), with all real coefficients, there is a charge conjugation symmetry C

$$\begin{aligned} C\phi C &= \bar{\phi}, \\ C\psi C &= \bar{\psi}, \end{aligned} \quad (46)$$

which is not spontaneously broken by the kink solution (38). Therefore, the following excited non-BPS states are degenerate:

$$|n\rangle, q^\dagger |n\rangle, Cq^\dagger |n\rangle, C|n\rangle. \quad (47)$$

The fermion charge assignments are, naturally, different from those we used in (4). Indeed, due to the fact that

$$CFC = -F, \quad (48)$$

the contribution to the index of the multiplet (47)

$$f(-1)^f + (f+1)(-1)^{f+1} - f(-1)^{-f} - (f+1)(-1)^{-f-1} = (-1)^{-f} - (-1)^f \quad (49)$$

does not vanish unless $f = 1$. Only if one adds the multiplet (47) to the degenerate one, with the fermion state on the zero-energy level⁶, does one get the overall zero contribution.

⁶For which the fermion number is $f+1$.

6 Multifield Wess–Zumino models

Our consideration can be easily generalized for the case of more than one field. Let us briefly sketch the procedure focusing on the bosonic fields. For n fields there is a contribution to the Hamiltonian of the form

$$\mathcal{H} - \zeta^{00} \supset (\partial_z \phi_i - \bar{\mathcal{W}}_i) (\partial_z \bar{\phi}_i - \mathcal{W}_i), \quad (50)$$

where

$$\mathcal{W}_i = \frac{\partial \mathcal{W}}{\partial \phi_i}, \quad i = 1, 2, \dots, n.$$

Therefore, the kink solution satisfies n equations

$$\partial_z \phi_i^k - \bar{\mathcal{W}}_i(\bar{\phi}^k) = 0. \quad (51)$$

Upon expansion around the kink the Hamiltonian becomes

$$(\mathcal{H} - \zeta^{00})_{\text{quad}} \supset \frac{1}{2} (\bar{\chi}, \chi) P^2 \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}, \quad (52)$$

where now χ and $\bar{\chi}$ are the columns of n elements

$$\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix}, \quad \bar{\chi} = \begin{pmatrix} \bar{\chi}_1 \\ \vdots \\ \bar{\chi}_n \end{pmatrix}, \quad (53)$$

and the operator P has the form

$$P = \begin{pmatrix} i\delta_{ij}\partial_z & -i\bar{\mathcal{W}}_{ij} \\ i\mathcal{W}_{ij} & -i\delta_{ij}\partial_z \end{pmatrix}, \quad (54)$$

with self-evident notation.

In order to discretize the spectrum we have to impose a boundary conditions for each field.⁷ We will act in the same way as before. We introduce the $2n$ -dimensional complex space \mathbb{C}^{2n} with a scalar product

$$(w, z) = w^\dagger z. \quad (55)$$

For the n -dimensional subspace consisting of all points (z, \bar{z}) , there is an induced scalar product

$$(w, z) = w^\dagger z + z^\dagger w, \quad (56)$$

⁷For n complex fields we have to impose $2n$ boundary conditions. For one field we had two.

which is real and has a usual form if one introduces real coordinates

$$z_i = \frac{x_i + iy_i}{\sqrt{2}},$$

namely,

$$(w, z) = u_i x_i + v_i y_i. \quad (57)$$

The kink can be represented as a line in this $2n$ -dimensional hypersurface,

$$\gamma_n^k = (\partial_z \phi_1^k, \dots, \partial_z \phi_n^k, \partial_z \bar{\phi}_1^k, \dots, \partial_z \bar{\phi}_n^k). \quad (58)$$

Then the boundary conditions are in fact the orthogonality conditions between the norms to the curve γ_n^k and the fluctuations.

7 A general perspective

Now, after we finished our box construction, we would like to discuss issues common to the Witten and CFIV indices from a more general standpoint.

The statement that the Witten index does not depend on the continuous deformations of the superpotential is a mathematically rigorous assertion. However, from the physics standpoint this assertion should be qualified. Indeed, it may well happen that under continuous deformations of the superpotential a supersymmetric vacuum (or vacua) of the theory run away to infinity in the space of fields, while a nonsupersymmetric minimum remains near the origin. This means that the supersymmetric vacuum decouples from the physical Hilbert space implying a change in the Witten index evaluated in the physical Hilbert space. The most well-known example of this type is the Intriligator–Thomas–Izawa–Yanagida (ITIY) model [8]: an $SU(2)$ super-Yang–Mills theory with a judiciously chosen matter sector (for a review see [9]).

The model is nonchiral, therefore, the Witten index equals two. Nevertheless, supersymmetry is dynamically broken, i.e. effectively the Witten index vanishes!

In the ITIY model we have four “quark” superfields Q_f^α , each is a color doublet ($\alpha = 1, 2$ and $f = 1, 2, 3, 4$). In addition to the quark superfields Q_f^α , six color-singlet chiral superfields $S^{fg} = -S^{gf}$ are introduced. Their interaction with $Q_{\alpha f}$ is due to the superpotential,

$$\mathcal{W} = \frac{h}{2} S^{fg} Q_f^\alpha Q_g^\beta \epsilon_{\alpha\beta} + m S^2. \quad (59)$$

Two supersymmetric vacua were found [8] at

$$S = \pm \text{const } h m^{-1} \Lambda^2, \quad (60)$$

in full accord with Witten's index. However, in the limit $m \rightarrow 0$, when the second term in (59) disappears (certainly a smooth allowed deformation of the superpotential) these supersymmetric vacua escape to infinity in the space of fields. A non-supersymmetric vacuum survives at a finite distance from the origin in the space of fields [8]. From the physical point of view, in passing from $m \neq 0$ to $m = 0$ the Witten index jumps by two units.

The statement that the CFIV index is independent of the continuous deformations of the Kähler potential but depends on deformations of the superpotential, being mathematically accurate, must be qualified too. Indeed, if we have a short kink supermultiplet, and the superpotential parameters are not in the immediate vicinity of the curves (walls) of marginal stability, small variations of the superpotential cannot make a long supermultiplet out of the short one. Only if we change the parameters in such a way that we touch the curves (walls) of marginal stability, the missing states (needed to make a long supermultiplet from the short supermultiplet) come from the spatial infinity (now we mean not the space of fields, but just the z axis), see e.g. [10].

When we introduce a large box, strictly speaking, before taking the limit $L \rightarrow \infty$, we do not have spatial infinities. If we include the edges of the box into consideration and will not discriminate between the states localized on the kink and those localized on the box edges, all supermultiplets will become long, and the CFIV index of this expanded system will vanish regardless of which side of the curve (wall) of marginal stability we are.

Therefore, in both cases discussed above there is a subtlety associated with the run-away situations: either in the space of fields or in the configurational space. It is desirable to make the formal index analysis “know” about possible run-aways, in the most general form. In our problem we managed to avoid this issue by imposing special boundary conditions, a construction which is obviously not general.

8 Conclusions

Our basic idea is straightforward. Discretizing excitation spectrum while preserving enough supersymmetry, along with the BPS soliton with its moduli, allows us to achieve the nonzero mode cancelation in the CFIV index level-by-level. With the choice of the boundary conditions as given above in our paper, supersymmetry is manifest. The mode degeneracy appears

in much the same way as in the problem of instantons [11]. As a result the calculation of the CFIV index reduces to finding the contribution only from the ground state. Therefore, the index can be used to count the number of short multiplets in the theory.

In general, introducing boundaries or certain conditions far from the kink core may change the solution. But physically these possible changes do not affect the kink core per se, but may add (or subtract) “junk” at the boundaries which has to be eliminated from any physically sensible kink analysis anyway. The box we suggest is subtle, no “junk” sticks to its edges.

Assume we impose *ad hoc* boundary conditions in the kink sector which need *not* maintain supersymmetry. We will require, however, that they preserve the BPS kink and discretize the excitation spectrum, creating a mass gap $1/L$. Then the calculation of the CFIV index (defined in the normalization we use) should produce an integer result coinciding with ours in the limit when we first fix L , then tend $\beta \rightarrow \infty$ and only at the very end allow L to become infinite. And it does (see Sect. 4 in [2] which can be adjusted to yield this result). If one is not careful, one can first take $L \rightarrow \infty$, which makes the spectrum continuous and adds to the index some nonvanishing infrared excitation contributions with energies lower than $1/\beta$. However, the limits $L \rightarrow \infty$ and $\beta \rightarrow \infty$ are not interchangeable.

The procedure we suggest seems natural in view of the fact that in some instances the calculation of a nonvanishing CFIV index for a given kink is essentially the same as the calculation of a nonvanishing Witten index in supersymmetric quantum mechanics on the kink world line. One of simple examples of this type is provided by kinks in the $\mathcal{N} = (2, 2)$ CP(1) model with a (large) twisted mass. It is discussed in detail e.g. in [12].

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Appendices

A More on the boundary conditions

A.1 Trivial vacuum

In this section we illustrate how the boundary conditions (19) and (20) or their linear analogs (25) work in the case of topologically trivial vacuum. At the same time it is evident that similar situation occurs even for nontrivial background for sufficiently high excitations. It is enough to consider the eigenfunctions of the operator P^2 (22) satisfying the boundary conditions (28). For the case at hand the operator P^2 has the following form

$$P^2 = -\partial_z^2 \mathbb{1}. \quad (61)$$

To further simplify things we take a from (19) and (20) to be real (for high excitations around kink configuration it is always possible to do by simple phase rotation). The general solution for the equation

$$-\partial_z^2 \begin{pmatrix} f \\ \bar{f} \end{pmatrix} = \omega^2 \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \quad (62)$$

with boundary conditions

$$\begin{aligned} \text{Im} f|_{\pm \frac{L}{2}} &= 0, \\ \partial_z \text{Re} f|_{\pm \frac{L}{2}} &= 0. \end{aligned} \quad (63)$$

is given by

$$f_n = A \cos \omega_n \left(z + \frac{L}{2} \right) + iB \sin \omega_n \left(z + \frac{L}{2} \right), \quad (64)$$

with A and B being real, while $\omega_n = \frac{\pi n}{L}$. As a result one finds that the functions having the properties (27) are given by the following expressions

$$\begin{aligned} f_n^1 &= \frac{1}{\sqrt{2L}} e^{-i\omega_n(z+\frac{L}{2})}, \\ f_n^2 &= \frac{1}{\sqrt{2L}} e^{i\omega_n(z+\frac{L}{2})}. \end{aligned} \quad (65)$$

A.2 Beyond quadratic approximation

We have shown before that the boundary conditions (19) and (20) are exactly the ones described in Introduction, namely, such boundary conditions discretize the spectrum while preserving the BPS soliton with its modules and half of the supersymmetry. The valid question is, whether such boundary conditions can be generalized for arbitrary order in perturbation theory. To answer this we note that the boundary conditions (19) and (20) preserve half of the supersymmetry and BPS solution in arbitrary order, while if taken with $a = \partial_z \phi_k$ they do not respect the fermionic zero mode beyond quadratic approximation. Indeed, the zero mode satisfies the following operator equation

$$\left\langle \begin{pmatrix} \partial_z & -\bar{\mathcal{W}}'' \\ \mathcal{W}'' & -\partial_z \end{pmatrix} \begin{pmatrix} \psi_R \\ \bar{\psi}_L \end{pmatrix} \right\rangle = 0, \quad (66)$$

where the average is taken with respect to vacuum state (soliton) plus the fermionic mode

$$\langle \dots \rangle \equiv \langle \text{sol}, f | \dots | \text{sol}, f \rangle. \quad (67)$$

Therefore, the second set of boundary conditions (20) is automatically satisfied for the fermionic zero mode. However, we know that the zero mode is related to the BPS soliton profile through the supersymmetry transformations⁸

$$\begin{aligned} \langle \delta \psi_R \rangle &= \bar{\eta} \partial_z \langle \phi \rangle, \\ \langle \delta \psi_L \rangle &= \eta \partial_z \langle \phi \rangle. \end{aligned} \quad (68)$$

Hence, it does not satisfies the (19) with $a = \partial_z \phi_k$, but rather with $a = \partial_z \langle \phi \rangle$. The profile $\langle \phi \rangle$ can be found order by order in perturbation theory. Therefore, we have found the necessary boundary conditions for arbitrary order in perturbation theory.

A.3 Dependent boundary conditions

Consider the system of two first order linear differential equations on the interval $z \in [0, L]$

$$\begin{aligned} f_1' &= -\omega f_2, \\ f_2' &= \omega f_1. \end{aligned} \quad (69)$$

⁸The leading order version of this relations is (13).

In order to make the spectrum discrete one has to impose two independent boundary conditions. The following boundary conditions are possible, since they are independent

$$\begin{aligned} f_2(0) &= f_2(L) = 0, \\ \text{or} \\ f_2(0) &= f_1'(L) = 0. \end{aligned} \tag{70}$$

While the boundary conditions of the form

$$f_2(0) = f_1'(0) = 0 \tag{71}$$

are not independent and therefore can not be used to discretize the spectrum. Suppose we choose the boundary conditions given in the first line of (69). It is clear that adding another boundary conditions like

$$f_1'(0) = f_1'(L) = 0 \tag{72}$$

does not change the spectrum, since those boundary conditions follow from the first ones and the equations.

B Interpolating between arbitrary vacua

For a general parameter

$$\gamma = \frac{\Delta\bar{\mathcal{W}}}{|\Delta\mathcal{W}|} = e^{-2i\alpha}, \tag{B.1}$$

after redefining the fermionic fields

$$\psi_{R,L} \rightarrow e^{-i\alpha} \psi_{R,L}. \tag{B.2}$$

The Hamiltonian can be put in the form

$$\begin{aligned} \mathcal{H} &= \partial_0 \bar{\phi} \partial_0 \phi + (\partial_z \phi - \bar{\gamma} \bar{\mathcal{W}}') (\partial_z \bar{\phi} - \gamma \mathcal{W}') + \partial_z (\gamma \mathcal{W} + \bar{\gamma} \bar{\mathcal{W}}) \\ &+ \frac{i}{2} (\bar{\psi}_L \partial_z \psi_L - \bar{\psi}_R \partial_z \psi_R - \partial_z \bar{\psi}_L \psi_L + \partial_z \bar{\psi}_R \psi_R) + i\gamma \mathcal{W}'' \psi_R \psi_L + i\bar{\gamma} \bar{\mathcal{W}}'' \bar{\psi}_R \bar{\psi}_L, \end{aligned} \tag{B.3}$$

while the supertransformations upon the substitution

$$\varepsilon_{R,L} \rightarrow e^{i\alpha} \varepsilon_{R,L} \tag{B.4}$$

become

$$\begin{aligned}
\delta\phi &= i\sqrt{2}(\varepsilon_L\psi_R + \varepsilon_R\psi_L), \\
\delta\psi_L &= -\sqrt{2}\bar{\varepsilon}_R\partial_L\phi + \sqrt{2}\varepsilon_L\bar{\gamma}\bar{\mathcal{W}}', \\
\delta\psi_R &= -\sqrt{2}\bar{\varepsilon}_L\partial_R\phi - \sqrt{2}\varepsilon_R\bar{\gamma}\bar{\mathcal{W}}'.
\end{aligned} \tag{B.5}$$

Therefore, the problem is reduced to the previously solved problem with superpotential $\gamma\mathcal{W}$.

References

- [1] E. Witten, Nucl. Phys. **B202**, 253 (1982).
- [2] S. Cecotti, P. Fendley, K. A. Intriligator, and C. Vafa, Nucl. Phys. **B386**, 405 (1992). [hep-th/9204102].
- [3] A. V. Smilga, Nucl. Phys. **B266**, 45-57 (1986).
- [4] M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, Phys. Rev. **D59**, 045016 (1999). [hep-th/9810068].
- [5] K. Hori, C. Vafa, *Mirror symmetry*, hep-th/0002222.
- [6] P. Fendley, *Exact information in $\mathcal{N} = 2$ theories*, hep-th/9305124.
- [7] R. Jackiw, C. Rebbi, Phys. Rev. **D13**, 3398-3409 (1976).
- [8] K. A. Intriligator, S. D. Thomas, Nucl. Phys. **B473**, 121-142 (1996) [hep-th/9603158]; K. -I. Izawa, T. Yanagida, Prog. Theor. Phys. **95**, 829-830 (1996) [hep-th/9602180].
- [9] M. Shifman, A. Vainshtein, *Instantons versus supersymmetry: Fifteen years later*, in M. Shifman, *ITEP lectures on particle physics and field theory* (World Scientific, Singapore, 1999), Vol. 2, Section 6.3 [hep-th/9902018].
- [10] A. Ritz, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, Phys. Rev. **D63**, 065018 (2001) [arXiv:hep-th/0006028 [hep-th]].
- [11] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. **B229**, 381 (1983).
- [12] M. Shifman and A. Yung, *Supersymmetric Solitons*, (Cambridge University Press, 2009).